Unbounded Axisymmetric FEM Formulation for Static Fields

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This paper describes an unbounded approach for solving the Poisson equation by the Finite Element Method (FEM). With domain mapping, it is unnecessary to truncate the domain at an arbitrary distance, where the potential is assumed negligible. The formulation herein has applications for static fields, has a simple implementation, and can handle infinite heterogeneous domains, which is a limitation of several FEM software. Such an approach raises the simulation precision and can save computational resources.

Index Terms—Electrokinetics, Electromagnetic fields, Electrostatics, Finite element analysis, Magnetostatics.

I. INTRODUCTION

POISSON’S equation can describe electromagnetic potentials in 2D, with suitable boundary conditions. This equation is generalized as:

\[ \nabla \cdot k \nabla \Theta + Q = 0 \]  \hspace{1cm} (1)

The parameter \( \Theta \) assumes \( V \), electric potential, for electrostatic and electrokinetic studies. Such a value is \( A \), magnetic vector potential normal component, for magnetostatic studies. The parameter \( k \) is permittivity \( \varepsilon \) for electrostatic studies, conductivity \( \sigma \) for electrokinetics, relucitivity \( \nu = 1/\mu \) for magnetostatic or a tensor reluctivity for permanent magnets studies. The parameter \( Q \) is charge density \( \rho \) for electrostatic studies or null for electrokinetics. It also assumes the current density \( J \) for magnetostatics.

This paper presents a domain mapping for the elements in the outer limits of a bidimensional domain, where the source term \( Q \) is supposed null for every study. We impose typical boundary conditions at border nodes. However, the mapped elements have an anisotropic behavior, and the material property has a spatial dependency.

II. FINITE ELEMENT MODELLING

The system’s governing equation is (1) in the \( x, y, z \) coordinate system with the usual Dirichlet and Neumann boundary conditions. The potential is exact at the Dirichlet boundary, making the variation and residual null. A proper choice of the Neumann boundary’s weight function results in automatic satisfaction of this boundary condition.

The FEM is a closed domain method. However, the following sections describe the unbounded approach based on domain mapping, thus being unnecessary to truncate the domain at an arbitrary distance where the potential is assumed negligible [2].

A. The unbounded approach

After applying of the proper weigh functions, the integral equation on \( \Omega \) obtained by the Weighted Residual Method is

\[ \int_{\Omega} kD_{xyz}^T W_0 D_{xyz} \Theta^e \, dxdydz = 0 \]  \hspace{1cm} (2)

Where \( W_0 \) is the associated weight, \( e \) is a subscript for the element, and the Del operator (>\( \nabla \)) is \( D_{xyz} = \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right] \). In a transformed coordinate system \((u, v, w)\), the analogous operator is \( D_{uvw} = \left[ \frac{\partial}{\partial u} \frac{\partial}{\partial v} \frac{\partial}{\partial w} \right] \). The Jacobian matrix \( J \) relates both operators as

\[ D_{xyz} = JD_{uvw} \quad \text{and} \quad dxdydz = |J^{-1}|dudvdw \]  \hspace{1cm} (3)

By including (3) into (2), we obtain

\[ \int_{\Omega^e} D_{uvw}^T W_0^e [k'] D_{uvw} \Theta^e \, du dv dw = 0 \]  \hspace{1cm} (4)

where

\[ [k'] = J^T k J |J^{-1}| \]  \hspace{1cm} (5)

Therefore, the FEM modeling for the mapped domain is equivalent to the solving of Poisson’s equation (1) in an anisotropic material \([k']\) described as a tensor, with the usual boundary conditions and the Del operator in the transformed coordinates.

B. The Jacobian matrix

This paper aims at the axisymmetric problem, but such mapping on planar symmetry is analogous, with minimal modification on the formulation. First, to describe the symmetric situation, we need only two variables \( u \) and \( v \). We relate the transformed coordinates with the \( r \) and \( z \) coordinates of an axisymmetrical cylindrical plane.

<table>
<thead>
<tr>
<th>Region</th>
<th>( r )</th>
<th>( \frac{z}{z_0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Region I</td>
<td>( r_m + r_f (r_f - r_m) )</td>
<td>( z_m + z_f (z_f - z_m) )</td>
</tr>
<tr>
<td>Region II</td>
<td>( r_m + r_f (r_f - r_m) )</td>
<td>( z_m + z_f (z_f - z_m) )</td>
</tr>
<tr>
<td>Region III</td>
<td>( r_m + r_f (r_f - r_m) )</td>
<td>( z_m + z_f (z_f - z_m) )</td>
</tr>
</tbody>
</table>

There are four possible mappings. The first is when both \( r \) and \( z \) are mapped, which is the most general case. But there are the cases where only \( r \) is mapped, only \( z \), and the unmapped case. Table II shows the variable transformations for every possible situation, the subscript \( f \) is the limit of the unmapped domain, and \( m \) is the limit for the mapped domain concerning the mapped variables.
Region IV is the most general region, so that we will develop our mapping for this region, and the other solutions will be developed based on that region. The mapped domains extend to infinity, i.e., \( r \in [r_f, \infty) \) and \( z \in [z_f, \infty) \), the region II has a finite \( z \) and region III has a finite \( r \). Note that \( u,v \) coordinates refer to the unmapped domain coordinates, those regions are extended toward infinity, \( u \in [r_f,r_m] \) and \( v \in [z_f,z_m] \). The Jacobian matrix of the transformation is

\[
J = \begin{bmatrix}
\frac{\partial u}{\partial r} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial r} & \frac{\partial v}{\partial z}
\end{bmatrix} = \begin{bmatrix}
M_r & 0 \\
0 & M_z
\end{bmatrix}
\tag{6}
\]

Assuming fixed \( r \) and \( z \) values inside each element, and being \( r_b \) and \( z_b \) the barycenter mapped coordinates, the \( M_r \) and \( M_z \) values are

\[
M_r = -r_f(r_f - r_m) \Rightarrow -r_f(r_f - r_m)/r_b^2
\tag{7}
\]

and

\[
M_z = -z_f(z_f - z_m) \Rightarrow -z_f(z_f - z_m)/z_b^2
\tag{8}
\]

Thus, the inverse of the Jacobian and its determinant are

\[
J^{-1} = \begin{bmatrix}
1/M_r & 0 \\
0 & 1/M_z
\end{bmatrix} \Rightarrow |J^{-1}| = \frac{1}{M_r M_z}
\tag{9}
\]

Including diagonal anisotropy for the unmapped domain, the transformed material property tensor is then

\[
[k'] = \begin{bmatrix}
M_r & 0 \\
0 & M_z
\end{bmatrix} \begin{bmatrix}
k_r & 0 \\
0 & k_z
\end{bmatrix} \begin{bmatrix}
M_r & 0 \\
0 & M_z
\end{bmatrix} \cdot \frac{1}{M_r M_z}
\tag{10}
\]

The solution for region IV and other regions are

\[
[k']^I = \begin{bmatrix}
k_r M_z & 0 \\
0 & k_z M_r
\end{bmatrix} \quad [k']^II = \begin{bmatrix}
k_r M_z & 0 \\
0 & k_z M_r
\end{bmatrix}
\]

\[
[k']^III = \begin{bmatrix}
k_r & 0 \\
0 & k_z
\end{bmatrix}
\]

If the material of the original problem is isotropic, we have \( k_r = k_z = k \). Note region IV is a general solution, and we can use this for every domain by assuming \( M_r \) or \( M_z \) equals unity at that domain.

**C. The Element Equation**

We here assume the mapped domains are source-free regions. The integral equation is obtained by the Weighted Residual Method applied to each element in the transformed domain. By using the Galerkin technique to \([1]\), we have the global equation

\[
\sum_{j=1}^{nn} \left[ \sum_{i=1}^{nn} 2\pi u_{ib} \int_{S'} N_i^j \cdot [k'] \nabla N_i^j \Theta_i^b \, dS' \right] = 0
\tag{11}
\]

where \( nn \) the number of nodes, \( u_{ib} \) is the barycenter coordinate for the elements \( u \) variable (original coordinates), and \( S' \) is a bidimensional domain. The element equation for linear shape functions

\[
N_i^j = \frac{1}{2\Delta} (a_i + b_i r + c_i z) \quad i = 1, 2, 3
\tag{12}
\]

being \( \Delta \) the area of the element is then

\[
K_{ij} = \frac{2\pi u_{ib}}{4\Delta} \left( k_r \frac{M_r}{M_z} b_i b_j + k_z \frac{M_z}{M_r} c_i c_j \right)
\tag{13}
\]

The global system \( [K][\Theta] = 0 \) is then obtained by the assembling process, which is modified to include Dirichlet boundary conditions \([3]\).

**III. Example Application**

In this section, we introduce an example application concerning an electrokinetic model. The rod has a 3 m length, ¾” diameter, embedded in the soil of 0.01 S/m conductivity, and a fixed voltage as in \([4]\) and the grounding resistance computed at postprocessing. The top interface has a Neumann condition to simulate the air, such as the left interface for the axial symmetry. The other interfaces have Dirichlet boundary conditions that are the reference for the potential at infinity.

![Fig. 1. Grounding rod with fixed potential embedded in a semi-infinite soil.](image)

This problem has a known analytical solution which is 32.57 \( \Omega \). Figure 1 shows the equipotentials for every domain of the simulated mesh. With a domain truncation, we achieve 31.42 \( \Omega \), which is an error of 3.53%. With the unbounded approach, we achieve 32.54 \( \Omega \), a deviation of only 0.11% using the same mesh of the previous result and with the same computational effort.

**References**


