Construction Principles of Electromagneto-Quasistatic Darwin Model Field Formulations

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Electromagneto-quasistatic (EMQS) fields, where capacitive, resistive, and inductive effects are considered in the absence of radiation effects, can be modelled using the Darwin-Ampère equation. Here, a systematic approach is proposed for deriving distinctly gauged scalar-vector potential formulations for EMQS fields. In view of the proposed methodology, the resulting EMQS field models not only cover established Darwin-type formulations, but also introduce novel formulations. Numerical experiments show the validity of such EMQS approximations in comparison to full Maxwell field simulations.

Index Terms—Computational electromagnetics, Darwin-type fields, electromagneto-quasistatic fields, time-domain analysis.

I. INTRODUCTION

The simulation of electromagnetic fields in the quasistatic regime, where resistive, inductive, and capacitive effects need to be considered while neglecting radiation effects, is an active field of research. For instance, such approximations are required in power electric circuits and coils, where both the electric and the magnetic energy density posses comparable magnitudes, and as a result, the use of merely electro- or magneto-quasistatic field models are inaccurate. For this reason, electromagneto-quasistatic (EMQS) field approximations \cite{[1]–[5]} to the full set of Maxwell equations, related to the original Darwin model for quasistatic field approximations \cite{[1]–[5]} to the full set of Maxwell equations, related to the magneto-quasistatic field models are inaccurate. For this reason, electromagneto-quasistatic (EMQS) field approximations \cite{[1]–[5]} to the full set of Maxwell equations, related to the original Darwin model for quasistatic field approximations for moving plasmas \cite{[6]–[8]}, have recently become of interest. Such Darwin-(type) EMQS field models are based on the reformulation of Ampère’s equation in terms of a magnetic vector potential $A$ and an electric scalar potential $\varphi$, while wave propagation is disregarded by neglecting the term $\varepsilon \partial_t A = \varepsilon \partial_t \partial_t A$, which is related to the rotational parts of the displacement current densities. The resulting so-called Darwin-Ampère continuity equation is

$$\text{curl}(\nu \text{curl} A) + \kappa \partial_t A + \kappa \text{grad} \varphi + \varepsilon \text{grad} \partial_t \varphi = J_S,$$ \hspace{0.5cm} (1)

where $\nu$, $\kappa$, and $\varepsilon$ are the reluctance, conductivity, and permittivity functions.

Provided that the magnetic vector potential is not unique, \cite{[1]} requires an additional gauge equation. Here, it is demonstrated that gauging equations may stem from general construction principles based on partial time-discretization of the divergence of the Maxwell continuity equation

$$\text{div} \left( \kappa \partial_t A + \kappa \text{grad} \varphi + \varepsilon \partial_t A + \varepsilon \text{grad} \partial_t \varphi \right) = \text{div} J_S.$$ \hspace{0.5cm} (2)

II. CONTINUITY GAUGE EQUATIONS

In \cite{[9]}, a class of Darwin approximations is based on the exchange of individual time derivative operators $\partial_t$ by a finite difference operator $d_k$, such as

$$\partial_t f \approx d_k f := \alpha_k \cdot (f^{n+1} - f^n)/\Delta t,$$ \hspace{0.5cm} (3)

which depends on the discrete time step length $\Delta t$, two or more evaluations of $f^n = f(t^n)$ and a scalar factor $\alpha_k$ that determines an implicit time integration scheme. These approximations shall be chosen so that the time integration scheme is stable and the system matrices enable the usage of efficient solvers e.g. because of their symmetry. Such time discretizations result in a class of Darwin models with gauge equations of the form

$$\text{div} \left( \kappa d_1 A + \kappa \text{grad} \varphi + \varepsilon d_2 d_3 A + \varepsilon \text{grad} d_4 \varphi \right) = \text{div} J_S.$$ \hspace{0.5cm} (4)

Formulations based on \cite{[2]} that keep the second-order time operator $\partial_{tt} A$ can be discretized directly with Newmark-beta type time integration schemes alongside time discrete schemes for the first order derivatives in \cite{[1]}. An alternative to \cite{[2]} is used in the Darwin formulations of \cite{[2] and [4]}, where $d_2 d_3 A$ is replaced by $\frac{\Delta \varphi}{\Delta t}$ $d_3 A$ and a common time discretization procedure with $d_1 = d_3 = d_4$. Similarly, a Darwin model in \cite{[3]} replaces the expression $\text{div} \left( d_2 d_3 A + \text{grad} \partial_t \varphi \right)$ with $\frac{\Delta \varphi}{\Delta t} \text{div} \left( d_3 A + \text{grad} \varphi \right)$ in \cite{[4]}.

Using lowest order Whitney finite elements or the finite integration technique \cite{[10]} on \cite{[1]}, where $C$, $G$ are discrete curl and gradient operators, $M_\sigma$, $M_\kappa$, and $M_\varepsilon$ := $M_\kappa + (\alpha/\Delta t) M_\varepsilon$ are the material matrices, and $a$, $\Phi$ are the vector and scalar potential degrees of freedom vectors, respectively, and suitably choosing discrete versions of \cite{[2]}, a symmetric algebraic system of equations of the form

$$\left[ \begin{array}{cc} \frac{\Delta \varphi}{\Delta t} M_\kappa + C^T M_\sigma C & M_\varepsilon G \\ G^T M_\sigma & \frac{\Delta \varphi}{\Delta t} G^T M_\varepsilon G \end{array} \right] \left[ \begin{array}{c} a \\ \Phi \end{array} \right] = \text{rhs}(a^n, \Phi^n, J_S^{n+1}, \Delta t)$$ \hspace{0.5cm} (5)

needs to be solved at each time step. Solutions at previous time steps and different semi-discrete versions of \cite{[3]} only appear in the right hand side vectors rhs of \cite{[3]}. 

III. INCOMPLETE CONTINUITY GAUGE EQUATIONS

A second approach to the derivation of gauge equations for a corresponding family of Darwin-type EMQS models lies in the omission of terms in \cite{[2]}, which in case of expression parts
with time derivatives corresponds to the choice of $\alpha_k = 0$ for some $k$ in the semi-time-discrete full Maxwell gauge equations. For these formulations, the resulting monolithic space and time discrete formulations of the form (5) may no longer have symmetric system matrices.

This approach also includes existing Darwin-(type) formulations: The Darwin EMQS field model in (1) eliminates the expression $\text{div}(\varepsilon \partial_t A)$ in (2), which yields the so called Darwin-continuity gauge equation

$$\text{div}(\kappa \partial_t A + \varepsilon \text{grad} \varphi + \varepsilon \text{grad} \varphi) = \text{div} J_S. \quad (6)$$

The lack of an expressions $\text{div}(\varepsilon \partial_t A)$ in (6) (similar to a choice $\alpha_2 = \alpha_3 = 0$), corresponds to eliminating an intrinsic Coulomb gauge to the irrotational parts of the magnetic vector potential in the non-conductive regions and thus requires the introduction of an additional regularization of the Darwin-Ampère equation by either introducing a small artificial conductivity value in the non-conductive regions or by an additionally grad-div-regularization (1).

Neglecting $\text{div}(\kappa \partial_t A + \varepsilon \partial_t A)$ (corresponding to a choice $\alpha_1 = \alpha_3 = 0$) in (2) yields the electro-quasistatic continuity gauge equation

$$\text{div}(\kappa \varphi + \varepsilon \text{grad} \partial_t \varphi) = \text{div} J_S. \quad (7)$$

The Darwin model consisting of (1) and (7) can be discretized into a non-symmetric monolithic formulation. The formulation also allows a decoupled solution given by the sequential two-step Darwin model presented in (5). However, the specific use of (7) with the Darwin-continuity equation (2) implicitly given by (1) enforces the magneto-quasistatic continuity equation

$$\text{div}(\kappa \partial_t A) = 0. \quad (8)$$

As a result, this model does not describe inductive effects driving charge distributions in conductors resulting in electro-quasistatic fields as they occur e.g. in capacitor structures. Such effects, however, can be described with (5) or variants of the full Maxwell gauge equations.

IV. Numerical Results

The circuit depicted in Fig. 1 (2 cm) is supplied with a ramped sinusoidal excitation (10 MHz) and field quantities are computed using several EMQS time-domain models, with the trapezoidal rule for 2.5 periods, and are compared versus a full Maxwell frequency domain solver. Fig. 1 shows both the magnitudes of the electric field intensity and the magnetic flux density at $t = 225$ ns. In Fig. 2 the relative difference, with reference to full Maxwell field quantities, of the electric field intensity and the magnetic flux density are also depicted.

V. Conclusion

Electromagnetic quasistatic field formulations based on the Darwin-Ampère equation with various gauge equations have been presented. The full paper will add more details and numerical comparisons to full Maxwell solution schemes.

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REFERENCES