Determination of the maximum size of finite elements in eddy-current layers of rotating electrical machines

Serguei Maximov\textsuperscript{1}, David A. Aragon-Verduzco\textsuperscript{2}, Student, IEEE, Rafael Escarela-Perez\textsuperscript{3}, Senior, IEEE and Juan C. Olivares-Galvan\textsuperscript{3}, Senior, IEEE

\textsuperscript{1}Instituto Tecnologico de Morelia, Morelia, 58120, Mexico; on sabbatical leave: Universidad Autonoma Metropolitana-Azcapotzalco, 02200, Mexico City, Mexico
\textsuperscript{2}Universidad Autonoma Metropolitana-Azcapotzalco, 02200, Mexico City, Mexico
\textsuperscript{3}Universidad Autonoma Metropolitana-Azcapotzalco, 02200, Mexico City, Mexico

A proper determination of the maximum size of the elements, involved in the finite element analysis, is obtained for moving eddy-current layers of electromagnetic devices with moving components, such as electrical rotating machines. The problem of diffusion of electromagnetic field into a moving magnetic slab is solved analytically and numerically by employing the Galerkin method, with appropriately established boundary and initial conditions. A rigorous analysis of the analytical solution shows that the maximum size of elements, that do not lead to stiffness phenomenon, depends on both the velocity of the moving part and frequency of the incident electromagnetic field.

Index Terms—Finite element method, mathematical analysis, rotating electrical machines, stiffness phenomenon.

I. INTRODUCTION

Motion effects in electromagnetic devices with moving components have been previously studied using both analytical and finite element methods \cite{1}–\cite{4}. Movement leads to modification of the electromagnetic diffusion equation for the vector potential through the appearance of the term $v \times \nabla \times \mathbf{A}$ in the formula for the current density \cite{4}:

$$
\mathbf{J} = \sigma \left( -\frac{\partial \mathbf{A}}{\partial t} + v \times \nabla \times \mathbf{A} \right) + \mathbf{J}_s,
$$

where $\mathbf{J}_s$ is the external current density. The presence of the velocity term in the magnetic diffusion equation for the vector potential (1) may lead to a numerical instability to a degree depending upon the velocity magnitude \cite{5}. This phenomenon, known as singular perturbation, appears if the velocity term is large compared to the second order derivatives of vector potential. In this work, the boundary layer phenomenon is illustrated by considering an idealized one-dimensional limiting case in which the relative velocity of the moving component is constant. Previously obtained estimation of the maximum size of elements for FEM calculations is improved through a rigorous mathematical analysis.

II. PROBLEM

Let us consider a one-dimensional problems, where $\mathbf{A} = A(x, t) \mathbf{e}_y$ and $v = v \mathbf{e}_x$. Then, the magnetic diffusion problem, with the appropriate boundary and initial conditions, becomes:

$$
\begin{align*}
\frac{\partial^2 A}{\partial x^2} + \frac{\partial A}{\partial x} &= \frac{\partial^2 A}{\partial x^2} + \mu \sigma \frac{\partial A}{\partial t}, \\
A_c(0, t) &= e^{j\omega t}, \\
\left( v \frac{\partial A_c}{\partial x} - \frac{\partial A_c}{\partial t} \right) |_{x=0} &= 0 \quad \forall t, \\
A_c(x, 0) &= 0 \quad \text{for} \quad x > 0,
\end{align*}
$$

where $A(x, t) = \text{Re} A_c(x, t)$ and $m = \sigma \mu v$ is the magnetic Reynolds number. Problem (2) models the phenomenon of diffusion of an oscillating electromagnetic field into a moving magnetic slab. It can be solved applying the Laplace transform. The asymptotic solution is obtained in the limit $t \to 0$ as:

$$
A_c(x, t) = \frac{\sqrt{m^2 + j\omega \mu \sigma + \frac{m^2}{2}}}{2\sqrt{m^2 + j\omega \mu \sigma}} \times e^{j\omega t - e^{-\frac{m^2}{4}t}} e^{\frac{m^2}{4}x - \frac{m^2}{2} + j\omega \mu \sigma} + \frac{1}{2} e^{-\frac{m^2}{4} t + \frac{m^2}{2} x} \left( 1 - \frac{m \xi}{2} \right) \text{erfc} \left( \frac{x - \frac{m \xi}{2}}{\frac{m \xi}{2}} \right) + \mathcal{O} \left( \sqrt{t} \right),
$$

where $\mathcal{O}(\sqrt{t})$ refers to the Landau big-O-notation and $\text{erfc}(x)$ is the complementary error function. In order to estimate the maximum size of elements for FEM-analysis, the following approximation for the complementary error function \cite{6} can be used:

$$
\text{erfc}(x) \approx \frac{1}{\sqrt{\pi}} \frac{x}{1 + x^2} e^{-x^2}.
$$

Therefore, the approximate solution (3) can be represented in a more suitable form:

$$
\frac{\sqrt{m^2 + j\omega \mu \sigma + \frac{m^2}{2}}}{2\sqrt{m^2 + j\omega \mu \sigma}} \left( e^{j\omega t - e^{-\frac{m^2}{4}t}} \right) = a(t) e^{j\Psi(t)},
$$

where $\frac{m^2}{2} + j\omega \mu \sigma = \rho e^{j\theta}$. Here $\rho, \theta, \Psi(t) \in \mathbb{R}$ and $a(t)$ is the real amplitude. Thus, the real solution of Eq. (2) takes the form:

$$
A(x, t) = a(t) \exp \left\{ -x \left( \sqrt{\rho} \cos \frac{\theta}{2} - \frac{m}{2} \right) \right\} \times \cos \left( x \sqrt{\rho} \sin \frac{\theta}{2} - \Psi(t) \right).
$$
The maximum size of finite elements in the eddy-current layer can be estimated from the following condition:

\[ h \sup_{x>0} \left| \frac{\partial A}{\partial x} \right| < |A|. \]

Substitution of (5) into this condition yields:

\[ h < \frac{1}{m} \frac{2\sqrt{2}}{\sqrt{2} + \sqrt{1 + \sqrt{1 + \left( \frac{4\omega\mu\sigma}{m} \right)^2}}} \]

Thus, the new estimation (6) depends on both the velocity of the moving part and frequency of the incident electromagnetic field.

III. THE GALERKIN METHOD ANALYSIS

A numerical solution to Eq. (2) can be obtained using the Galerkin method in the form:

\[ A(x, t) = \sum_{n=1}^{N} A_n(t) W_n(x), \]

where \( W_n(x) \) is the standard weighting basis [4], \( N \) is the number of basis elements required to cover the domain. Eq. (2), with formula (7) taken into account, yields:

\[
\begin{align*}
\frac{1}{h} A_{n+1} - & \frac{2}{h} A_n + \frac{1}{h} A_{n-1} = - \frac{m}{2} A_{n-1} + \frac{m}{2} A_{n+1} \\
& + \mu \sigma \left( \frac{h}{6} \frac{dA_n}{dt} + \frac{2h}{3} \frac{dA_n}{dt} + \frac{6h}{6} \frac{dA_n}{dt} \right). 
\end{align*}
\]

(8)

Let us consider only steady state solution to Eq. (8) that can be represented in the form:

\[ A_n(t) = \text{Re} \left\{ \mathcal{A}_n e^{i\omega t} \right\}, \]

where \( \mathcal{A}_n \in \mathbb{C} \). Then, in steady state (8) becomes:

\[ \mathcal{A}_{n+1} = -p \mathcal{A}_n - q \mathcal{A}_{n-1}, \]

where the following notations have been introduced:

\[ p = -2 \frac{1 + j \omega \mu \sigma} {1 - \frac{m} {2} - j \omega \mu \sigma}, \quad q = 1 - \frac{m} {2} - j \omega \mu \sigma \]

(11)

Recurrent Eq. (10) can be solved using standard methods. The solution has the form:

\[ \mathcal{A}_n = (\mathcal{A}_1 - a \mathcal{A}_0) \frac{b^n}{b - a} + (\mathcal{A}_1 - b \mathcal{A}_0) \frac{a^n}{a - b}, \]

(12)

where \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) are constants, and \( a \) and \( b \) are solutions of the system of equations \( a + b = p \) and \( ab = q \). The following estimation comes from Eq. (11):

\[ |a| |b| = |q| = \sqrt{\left( 1 + \frac{m}{2} \right)^2 + \left( \omega \mu \sigma \frac{h^2}{n} \right)^2} > 1, \]

which means that either \( |a| > 1 \) or \( |b| > 1 \). Let be \( |b| > 1 \). Therefore, in order to avoid instability in solution (12), the condition \( 2 \mathcal{A}_1 - a \mathcal{A}_0 = 0 \) has to be accomplished. Substituting this condition into solution (12), we finally get:

\[ \mathcal{A}_n = \mathcal{A}_0 a^n. \]

Fig. 1 shows the comparison of solutions (3) and (13) for different sizes of finite elements. Case a) corresponds to finite elements with a size that satisfies condition (6), whereas in case b) condition (6) is not fulfilled. The parameters values taken for the simulations are: \( m \approx 377 m^{-1}, \sqrt{\omega \mu \sigma} = 238.4 m^{-1} \).

IV. CONCLUSION

An estimation for the maximum size of finite elements in eddy-current layers was obtained through a rigorous mathematical analysis of the magnetic field, penetrating into moving parts of electromagnetic devices. Unlike previous works, our analysis takes account of both the magnetic material movement and rapid magnetic field decrement in the boundary layer. Our results agree with the Galerkin method computations. Our 1D-results can be extended to the 2D and 3D cases, using the superposition principle.

REFERENCES